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Exercises: Charlotte Amann-Bartnick

Albert-Ludwigs-Universität Freiburg
Winter term 2025/2026

Model Theory
Sheet 10
Deadline: 08.01.**2026**, 2:30 pm.

Exercise 1 (9 points)**.**

Let $\lambda > \aleph_0$ be an infinite cardinal. In the first part of this exercise, we will construct a linear order without endpoints of cardinality strictly bigger than λ with a dense subset of size at most λ.

In order to do so, choose a cardinal μ smallest possible with $2^\mu > \lambda$ (why does such a μ exist?) and consider the set

$$P = \{f : \mu \to 2 \mid \text{ there is no } \alpha < \mu \text{ s.t. } f(\beta) = 1 \text{ for all } \beta > \alpha\}$$

a) Show that

$$f < g \Leftrightarrow \text{ there is an } \alpha < \mu \text{ such that } f\restriction_\alpha = g\restriction_\alpha \text{ and } f(\alpha) < g(\alpha)$$

defines a linear order on P without endpoints.

b) Prove that $Q = \{f \in P \mid \text{ there is an } \alpha < \mu \text{ such that } f(\beta) = 0 \text{ for all } \beta > \alpha\}$ is dense in P (with respect to the order topology on P whose basis is given by all open intervals).

c) Show that P has cardinality $2^\mu > \lambda$ whereas Q has cardinality at most λ.

We now consider the theory DLO of dense linear orders without endpoints in the language $\mathcal{L} = \{<\}$.

d) Show that for every infinite linear order I there is a model $\mathcal{M}$ of DLO with an increasing indiscernible sequence $(a_i)_{i\in I}$.

e) Using the previously defined ordering P, conclude that there is no infinite cardinal λ such that DLO is λ-stable.

Exercise 2 (6 points)**.**

Given an infinite $\mathcal{L}$-structure $\mathcal{M}$ and a subset A of M, recall that $\mathrm{acl}^{\mathcal{M}}(A)$ is the algebraic closure of A in $\mathcal{M}$ (see Sheet 8, Exercise 3).

a) Given an elementary map $F : A \to B$, for some subset B of M, show that F can be extended to an elementary map

$$\tilde{F} : \mathrm{acl}^{\mathcal{M}}(A) \to \mathrm{acl}^{\mathcal{M}}(B).$$

Hint: Exercise 4 of Sheet 6 + Zorn. You may use that $\mathrm{acl}^{\mathcal{M}}(\mathrm{acl}^{\mathcal{M}}(A)) = \mathrm{acl}^{\mathcal{M}}(A)$ (as was proven in the last exercise class).

b) Assume now that M is a saturated $\mathcal{L}$-structure. Give a short proof of the above whenever A and B have both cardinality strictly less than $|M|$.

(Please turn the page!)

The exercise sheets can be handed in in pairs. Submit them in the mailbox 3.19 in the basement of the Mathematical Institute.

Exercise 3. (5 points)

Let T be a totally transcendental complete theory of a group in a countable language $\mathcal{L}$ which contains the language $\{e, \cdot, ^{-1}\}$ of groups and $\mathcal{G}$ a saturated model of T. Recall that every subgroup H of G defines an equivalence relation

$$g_1 \sim_H g_2 \iff g_2^{-1} \cdot g_1 \in H.$$

The class of g_1 modulo $\sim_H$ is denoted by $g_1 \cdot H$ (and is called the H-coset of g_1).

a) Show that there is no descending chain of proper definable subgroups $H_1 \gneqq H_2 \gneqq H_2 \gneqq \ldots \gneqq H_n \gneqq \ldots$ of G.

   Hint: Trees.

b) Conclude from a) that every intersection of definable subgroups of G is definable.

c) For a subset A of G, the *centralizer* of A is the subgroup

$$C_G(A) = \{g \in G \mid g \cdot a = a \cdot g \text{ for all } a \text{ in } A\ \}.$$

   Show that $C_G(A)$ is always definable. If A is moreover invariant under all automorphisms of G with $|A| < |G|$, then conclude that $C_G(A)$ is definable without parameters.